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Letter Section

Conditional minimax estimates

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Abstract: In the present note a characterization of conditional minimax estimates for a parameter lying in a bounded interval is given under quite general conditions on the loss function. This result simplifies the computation of conditional minimax estimates.

Keywords: Conditional minimax estimate, bounded parameter interval, general loss function.

1. Introduction and notation

Let $\{P_\theta \mid \theta \in \Theta\}$ be a family of Borel probability measures on the sample space X, where $\Theta = [\underline{\theta}, \bar{\theta}] \subset \mathbb{R}$ is a compact interval. Let $\{f(\theta, \cdot) \mid \theta \in \Theta\}$ be a corresponding family of densities with respect to some σ-finite Borel measure. Suppose that the function $f(\cdot, x) : \Theta \to [0, \infty)$ is continuous for every $x \in X$. The parameter θ is to be estimated under a continuous loss function $s : \Theta \times A \to [0, \infty)$, where $A = \Theta$ denotes the action space. Assume that the function $s(\theta, \cdot)$ is decreasing on $[\underline{\theta}, \theta]$ and increasing on $[\theta, \bar{\theta}]$ for every $\theta \in \Theta$.

Subsequently, let the observation $x \in X$ be fixed. In the present note a conditional point of view is considered. Berger [1] writes: "The conditional approach to statistics is concerned with reporting data specific measures of accuracy. The overall performance of a procedure is deemed to be of (at most) secondary interest." (cf. [1, ch. 1.6.3, p.24]). In the following it is assumed that no prior information on the distribution of the unknown parameter θ is available besides the fact that θ is an element of the compact parameter interval Θ. Hence, no Bayesian analysis is possible and it seems to be reasonable to consider the following conditional minimax approach. An action a_x with

$$\max_{\theta \in \Theta} R_x(\theta, a_x) = \min_{a \in A} \max_{\theta \in \Theta} R_x(\theta, a)$$

will be called a conditional minimax estimate, where

$$R_x(\theta, a) = s(\theta, a) f(\theta, x)$$

for $(\theta, a) \in \Theta \times A$.

2. Characterization of conditional minimax estimates

Let M_l, $M_r : A \to [0, \infty)$ be defined by

$$M_l(a) = \max\{ R_x(\theta, a) \mid \theta \in [\underline{\theta}, a] \}$$

and

$$M_r(a) = \max\{ R_x(\theta, a) \mid \theta \in [a, \bar{\theta}] \},$$

respectively.

Theorem 1. *There exists an action a_x with $M_l(a_x) = M_r(a_x)$. Such an action a_x is a conditional minimax estimate.*

Proof. The function $R_x : \Theta \times A \to [0, \infty)$ is continuous. Hence, the functions M_l and M_r are continuous as well. A short calculation shows that M_l is increasing and that M_r is decreasing. Since $M_l(\underline{\theta}) \leqslant M_r(\underline{\theta})$ and $M_l(\bar{\theta}) \geqslant M_r(\bar{\theta})$ it follows that there exists an action a_x with $M_l(a_x) = M_r(a_x)$. Finally, the identity

$$\max_{\theta \in \Theta} R_x(\theta, a) = \max\{ M_l(a), M_r(a) \}$$

for $a \in A$ implies that

$$\max_{\theta \in \Theta} R_x(\theta, a_x) \leqslant \max_{\theta \in \Theta} R_x(\theta, a)$$

for $a \in A$, i.e., a_x is a conditional minimax estimate. □

Example 2. Suppose that a binomial probability $\theta \in [0, 1]$ is to be estimated after n observations under a loss function $s : [0, 1] \times [0, 1] \to [0, \infty)$ of the form

$$s(\theta, a) = \begin{cases} \alpha(a - \theta)^p & \text{for } \theta \leqslant a, \\ (\theta - a)^q & \text{for } \theta \geqslant a, \end{cases}$$

with fixed parameters α, p, $q > 0$. Hence, for $x \in \{0, 1, \ldots, n\}$

$$R_x(\theta, a) = \begin{cases} \alpha(a - \theta)^p \theta^x (1 - \theta)^{n-x} & \text{for } \theta \leqslant a, \\ (\theta - a)^q \theta^x (1 - \theta)^{n-x} & \text{for } \theta \geqslant a. \end{cases}$$

Some calculations show that for any $x \in \{0, 1, \ldots, n\}$ the functions M_l and M_r are given by $M_l(a) = R_x(\theta_l(a), a)$ and $M_r(a) = R_x(\theta_r(a), a)$, respectively, where

$$\theta_l(a) = \frac{na + p + x}{2(n + p)} - \left(\frac{(na + p + x)^2}{4(n + p)^2} - \frac{xa}{n + p} \right)^{1/2}$$

and

$$\theta_r(a) = \frac{na + q + x}{2(n + q)} + \left(\frac{(na + q + x)^2}{2(n + q)^2} - \frac{xa}{n + q} \right)^{1/2}$$

for $a \in [0, 1]$. Now, according to Theorem 1, the conditional minimax estimate a_x for $x \in \{0, 1, \ldots, n\}$ is the (unique) solution of the equation $M_l(a_x) = M_r(a_x)$.

Reference

[1] J.O. Berger, *Statistical Decision Theory and Bayesian Analysis* (Springer, New York, 2nd ed., 1985).